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LETTER TO THE EDITOR

A new percolation statistic with unusual properties

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Abstract. In bond percolation on a lattice with $p > p_c$, there are unoccupied bonds which, if occupied, would immediately join the backbone. In general, when such a bond is occupied, several 'tag end' bonds may join the backbone as well. We define n_i to be the number of bonds that would join the backbone if such a bond i were occupied and $\bar{n}$ to be the average of n_i taken over all such bonds on the lattice. We derive a formula for $\bar{n}$ and find that for all lattices, near p_c, $\bar{n} = \gamma_B p_c (p - p_c)^{-1}$, where γ_B is the exponent for the divergence of the backbone fraction. Identical formulae apply to site percolation.

Near above the percolation threshold in bond or site percolation on an infinite lattice, the percolation fraction P, the backbone fraction B and conductivity G obey relations of the form [1-3]

$$X = k_x (p - p_c)^{\gamma_x} \qquad X = P, B \text{ or } G. \tag{1}$$

Here p is the fraction of bonds (or sites) randomly occupied, p_c is the percolation threshold, γ_x is the exponent for X, thought to be identical for all networks of a given dimensionality [2-5], and k_x, the pre-exponential factor for the variable of interest, is determined by the given network geometry. Estimated values of γ_B, the backbone fraction exponent, and of p_c for various networks in site and bond percolation are listed in table 1. The backbone fraction is the fraction of bonds (or sites, in a site percolation problem) that could conduct current if a voltage drop were imposed across the network.

We are interested here in the voltage drop across *unoccupied* (insulating) bonds or sites on an infinite lattice above the percolation threshold. Our interest arises from the problem of mobilising foam lamellae in porous media that block the flow of gas as missing bonds block the flow of electricity [13]. Similar conductivity problems have been addressed: the maximum voltage drop across *occupied* bonds in the lattice above p_c [14-17] and the maximum voltage drop across *unoccupied* bonds near *below* p_c, where the occupied and unoccupied bonds are, respectively, conductors and insulators [14] or superconductors and conductors with finite resistivity [15].

Our problem differs from these in that an infinite cluster of finite conductivity exists, and we focus on missing bonds or sites that block clusters of dendritic or 'tag end' bonds (sites) from joining the backbone. In this letter we focus not on voltage drop *per se* but on cluster size, and not on the largest cluster, but on the average cluster; and we find that the average cluster size follows a particularly simple form of (1).

We proceed for a bond percolation problem, but the arguments are identical for site percolation. Above the percolation threshold there are unoccupied bonds which,

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Table 1. Percolation threshold and backbone exponents for 2D and 3D networks[a]: (*a*) 2D networks: $\gamma_B \sim 0.51 \pm 0.03$[b]; (*b*) 3D networks: $\gamma_B \sim 1.11 \pm 0.05$[b]; (*c*) Bethe tree[c]: $\gamma_B = 2$.

	Percolation threshold	
Network	Site percolation	Bond percolation
(*a*)		
Triangle	0.696 2	0.652 71
Square	0.592 75	0.500 0
Hexagon	0.500 0	0.347 29
Diamond	0.428	0.388
Voronoi	0.5[d]	0.332[d]
(*b*)		
Simple cubic	0.311 7	0.249 2
Body-centred cubic	0.245	0.178 5
BCC2[e]	0.168 6[f]	0.099 1[f]
Face-centred cubic	0.198	0.119
Voronoi	0.145 3[f]	0.082 2[f]
(*c*)		
Percolation threshold $= 1/(z-1)$		
$z \equiv$ coordination number		

[a] Unless noted, estimates are from [6].
[b] Reference [7]. See also [8–11].
[c] Reference [12].
[d] Reference [9].
[e] Body-centred cubic with bonds between second-nearest neighbours.
[f] Reference [10].

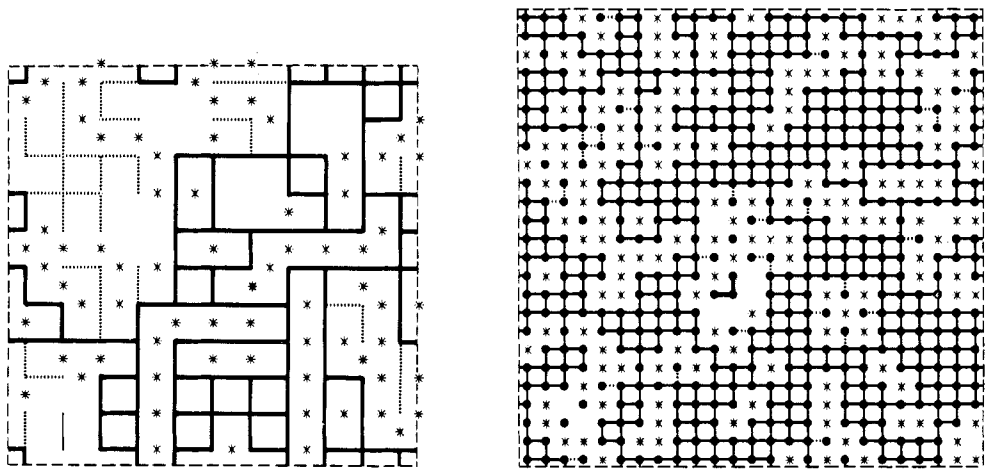

Figure 1. (*a*) Bond percolation. 143 bonds out of 242 are occupied: backbone (——) 111 bonds; dendritic (· · ·) 31 bonds; isolated (——) 1 bond. 99 bonds are unoccupied, of which 73 are b^* bonds (*). (*b*) Site percolation. 469 out of 625 sites are occupied: backbone (—●—) 431 sites; dendritic (· · · ● · · ·) 30 sites; isolated (—●—) 8 sites. 156 sites are unoccupied, of which 146 are b^* sites (*).

if occupied, would immediately join the backbone fraction. We call these bonds b^* bonds here. Examples are shown for percolation problems on a square lattice in figure 1. In general, if such a bond were occupied, not only it, but possibly several previously occupied but dendritic or 'tag end' bonds, would join the backbone. At a given p there are many b^* bonds, which we number i through N_b, on a lattice and associated with each is the pathway of size n_i of bonds that would be added to the backbone if that bond were occupied. (If only the b^* bond itself would join the backbone if it were occupied, $n_i = 1$). We derive here an expression for the average number of bonds in these pathways, $\bar{n}$, averaged over all b^* bonds in the network:

$$\bar{n} \equiv \left(\sum_{i=1}^{N_b} n_i \right) N_b^{-1}. \tag{2}$$

We derive a formula for $\bar{n}$ as follows. If an unoccupied bond is occupied, one of two events occurs. If the newly added bond does not join the backbone, then the backbone fraction is unchanged. If the newly occupied bond does join the backbone, the expected increase in the backbone population is $\bar{n}$. This is the same average $\bar{n}$ as defined by (2), because any b^* bond could be the next added with equal probability. Among the ensemble of sequences of adding bonds to reach a given configuration, any of the bonds could have been added last with equal probability; thus the probability that the last bond added joined the backbone is B/p, the fraction of all occupied bonds that are backbone bonds. Therefore

$$\frac{\mathrm{d}B}{\mathrm{d}p} = \left(1 - \frac{B}{p}\right) 0 + \frac{B}{p} \bar{n} = \frac{B}{p} \bar{n}. \tag{3}$$

Equation (3) implies that the fraction of unoccupied bonds that are b^* bonds equals the fraction of occupied bonds that are backbone bonds. In other words, the probability that the next bond added joins the backbone is the same as the probability that a bond removed reduces the backbone. This is roughly true for the small network realisations given in figure 1. It remains to confirm this implication, however, with Monte Carlo studies on large networks. Rearranging (3)

$$\bar{n} = p \frac{\mathrm{d} \ln B}{\mathrm{d}p}. \tag{4}$$

From (1), near p_c,

$$\lim_{p \to p_c} \bar{n} = p_c \gamma_B (p - p_c)^{-1}. \tag{5}$$

Equation (5) applies to site or bond percolation on lattices of any dimensionality. In contrast to P, B and G, the pre-exponential factor for $\bar{n}$ is a simple product of the exponent for the backbone fraction, thought to be identical for all lattices of given dimensionality [2-5], and the percolation threshold, which is tabulated for a variety of lattices. The exponent for $\bar{n}$ is (-1) for all lattices. Thus, using the quantities in table 1, one can determine $\bar{n}(p)$ near p_c for a wide variety of networks. Equations (4) and (5) for $\bar{n}$ are similar to those for the number of 'cutting' or 'red' bonds, without which the two ends of a cluster the size of the correlation length become disconnected [18-20]. Indeed, for that case as well the pre-exponential factor near p_c is a simple product of p_c and a critical exponent.

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